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OSCILLATIONS OF A RIGID BODY CONTAINING AN ELASTIC ELEMENT WITH DISTRIBUTED PARAMETERS†

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The motions of a hybrid (discrete-continual) system, consisting of a carrier rigid body and an elastic element with distributed parameters fastened to it are investigated. Two types of fastening are considered: (1) both ends are clamped, and (2) one of the ends is clamped while the other is free. A closed system of integro-differential equations is obtained which describes the state of the system under arbitrary initial conditions and forces applied to the rigid body. The perturbed motion of the rigid body in the case of a quasi-linear restoring force is investigated using asymptotic methods. The motions are studied both when there is internal resonance between the oscillations of the rigid body and the natural oscillations of the element, and when there are no such resonances. Qualitative effects are found.

1. FORMULATION OF THE PROBLEM

THE UNIAXIAL motions (along the OX axis) of a hybrid (discrete-continual) system, shown schematically in Fig. 1, are investigated. Here O is the origin of the fixed (inertial) reference frame, and s is the coordinate of the rigid body to which the movable system ox is connected. An elastic element with distributed parameters (a beam or a spring) is fastened to the carrier rigid body; the boundary conditions of the attachment will be discussed below. Suppose we are given arbitrary initial conditions of motion—the values of the coordinates and the velocity of the rigid body and also the distribution of the displacements and the velocities of an elastic section at a certain fixed instant of time t_0. It is required to investigate the motion of the rigid body and the relative motions of the element when a concentrated force P, external with respect to the system and dependent on the time t and on the phase variables s and $s^{\bullet}$, is applied to the carrier body.

In order not to complicate the description, we will assume that the density per unit length ρ and the rigidity to compression σ are constant (see the note at the end of Sec. 2). Assuming the deformations to be fairly small, we can write the following equations of state for the elastic element [1]

$$\rho u^{\bullet\bullet} = \sigma u'' - \rho s^{\bullet\bullet}, \quad u = u(t, x), \quad 0 < x < l \tag{1.1}$$

Here u is the relative elastic displacement of the section x, $0 \leqslant x \leqslant l$ at the instant t, $t \geqslant t_0$. We will consider two types of boundary conditions: (1) both ends of the elastic element are clamped

$$u(t, 0) = u(t, l) = 0, \qquad t \geqslant t_0 \tag{1.2}$$

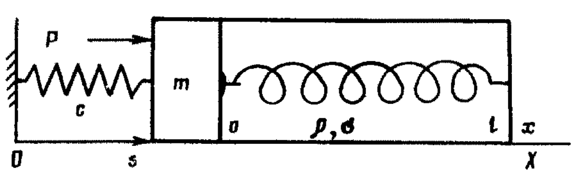

FIG. 1.

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and (2) one of the ends, say the left one, is clamped while the right one is free

$$u(t, 0) = u'(t, l) = 0, \quad t \geqslant t_0 \tag{1.3}$$

If the right end is clamped while the left one is free ($u'(t, 0) = u(t, l) = 0$), this case can be reduced to (1.3) by making the substitution $x = l - y$, $l \geqslant y \geqslant 0$. The problem of determining the unknown $u = u(t, x)$ will be closed if the motion $s = s(t)$—a twice continuously differentiable function, is known and the initial distributions $u(t_0, x)$, $u^{\cdot}(t_0, x)$ are specified

$$u(t_0, x) = f(x), \quad u^{\cdot}(t_0, x) = g(x), \quad 0 \leqslant x \leqslant l \tag{1.4}$$

where f and g are fairly smooth functions of a definite class [1, 2]. In the case of boundary conditions (1.2), the functions $f(x)$ and $g(x)$ must vanish when $x = 0, l$; for the case of conditions (1.3) they only vanish when $x = 0$.

The unknown $s = s(t)$ can be found by the simultaneous solution of the boundary value problem (1.1)–(1.3) with initial conditions (1.4) and Cauchy's problem for the integro-differential equation describing the motion of the centre of mass of the system (a Newton-type equation)

$$\begin{gathered} ms^{\cdot\cdot} + \int_0^l \rho[s^{\cdot\cdot} + u^{\cdot\cdot}(t, x)]\, dx = P(t, s, s^{\cdot}) \\ s(t_0) = s^0, \quad s^{\cdot}(t_0) = v^0 \end{gathered} \tag{1.5}$$

It is proposed to solve the combined problem (1.1)–(1.5) successively: first problem (1.1)–(1.4) is solved for an arbitrary displacement of the rigid body $s(t)$ from the required class of twice continuously differentiable functions. The function $u(t, x, [s^{\cdot\cdot}])$ represented in the form of the integral operator $s^{\cdot\cdot}(t)$ is substituted into (1.5) and a closed Cauchy integro-differential problem of the Volterra type in t [3.6] for the variable s is obtained, which is to be determined.

Using Eq. (1.1), the Cauchy integro-differential problem (1.5) can be reduced to a more-convenient form, not containing the integral with respect to x

$$\begin{gathered} ms^{\cdot\cdot} = P(t, s, s^{\cdot}) - \sigma u'(t, x, [s^{\cdot\cdot}])\big|_{x=0}^{x=l} \\ s(t_0) = s^0, \quad s^{\cdot}(t_0) = v^0 \end{gathered} \tag{1.6}$$

Hence, the equation of motion of the centre of mass of the hybrid system in the form (1.5) or (1.6) is then used to determine the coordinate $s(t)$, which defines the motion of the rigid body (the point o). If the solution $s(t)$ is constructed, the remaining characteristics of the motion of the whole system can be found fairly simply by using standard operations. Hence, our main attention will be devoted to deriving the Cauchy integro-differential problem for s in explicit form, and constructing and analysing the solution when a number of simplifying assumptions, which have a definite mechanical meaning, are satisfied.

Further, for convenience, we will change to dimensionless variables, arguments and parameters using the following formulae

$$\begin{gathered} \theta = \nu t, \quad \nu^2 = \sigma/(\rho l^2), \quad x = \xi l \\ s = \eta l, \quad u = zl, \quad \xi \in [0, 1] \end{gathered} \tag{1.7}$$

We will again denote derivatives with respect to the new arguments of time $\theta \geqslant \theta_0$ and coordinate ξ, $0 \leqslant \xi \leqslant 1$ by dots and primes, respectively. As a result, we obtain a combined problem in the unknowns $z = z(\theta, \xi)$ and $\eta - \eta(\theta)$

$$\begin{gathered} z^{\cdot\cdot} = z'' - \eta^{\cdot\cdot}, \quad 0 < \xi < 1 \\ 1)\ z(\theta, 0) = z(\theta, 1) = 0; \quad 2)\ z(\theta, 0) = z'('\theta, 1) = 0 \end{gathered} \tag{1.8}$$

$$z(\theta_0, \xi) = \varphi(\xi), \quad z^{\cdot}(\theta_0, \xi) = \psi(\xi)$$

$$\eta^{\cdot\cdot} = \Pi(\theta, \eta, \eta^{\cdot}) + \epsilon[z'(\theta, 0) - z'(\theta, 1)] \tag{1.9}$$

$$\eta(\theta_0) = \eta^0 \equiv s^0/l, \quad \eta^{\cdot}(\theta_0) = \nu^0 \equiv v^0/(\nu l), \quad \epsilon = \rho l/m$$

The functions φ, ψ and π are obtained from f, g and P, respectively, by making the change (1.7), while the parameter ϵ represents the effect of the elastic element on the motion of the carrier body. Other methods of changing to dimensionless variables are also possible, dictated by convenience and the nature of the problem. It is preferable, in our further investigations, to use the changes (1.7) to construct and analyse the solution of problems (1.8) and (1.9).

2. SOLUTION OF THE BOUNDARY VALUE PROBLEMS FOR KNOWN MOTION OF THE CARRIER BODY

The required distribution of elastic displacements $z(\theta, \xi)$, which satisfy boundary conditions 1 and 2 of problem (1.8) for the continuous function $\eta^{\cdot\cdot}(\theta)$, $\theta \in [\theta_0, \theta^*]$, $\theta^* \leqslant \infty$, can be constructed by standard methods of mathematical physics [1, 2]. As a result, we obtain the expressions

$$z(\theta, \xi) = \sum_{n=1}^{\infty} \Theta_n(\theta)\, \Xi_n(\xi), \qquad \Xi_n(\xi) = \sin\lambda_n \xi \tag{2.1}$$

$$1)\ \lambda_n = \pi n; \quad 2)\ \lambda_n = \tfrac{1}{2}\pi(2n-1), \qquad n \geqslant 1$$

Here $\{\lambda_n\}$, $\{\Xi_n(\xi)\}$ are systems of eigenvalues and eigenfunctions of the corresponding boundary value problems.

The Fourier coefficients $\Theta_n(\theta)$, $n \geqslant 1$ in (2.1) are obtained as the solutions of a denumerable set of Cauchy problems of the form

$$\Theta_n^{\cdot\cdot} + \nu_n^2 \Theta_n = -d_n \eta^{\cdot\cdot}, \quad \nu_n = \lambda_n, \quad n = 1, 2, \ldots \tag{2.2}$$

$$\Theta_n(\theta_0) = \varphi_n, \quad \Theta_n^{\cdot}(\theta_0) = \psi_n; \quad 1)\ \nu_n = \pi n, \quad 2)\ \nu_n = \tfrac{1}{2}\pi(2n-1)$$

Here φ_n, ψ_n $(n \geqslant 1)$ are the Fourier coefficients for the expansions of the functions $\varphi(\xi)$, $\psi(\xi)$ in the basis functions $\Xi_n(\xi)$, and ν_n are the eigenfrequencies $(\nu_n = \lambda_n)$. The coefficients d_n in (2.2) determine the sensitivity of each mode with respect to the external kinematic "action" $\eta^{\cdot\cdot}$. We have the following expressions

$$\varphi_n = 2\int_0^1 \varphi(\xi) \sin\lambda_n \xi\, d\xi, \quad \psi_n = 2\int_0^1 \psi(\xi) \sin\lambda_n \xi\, d\xi$$

$$d_n = 2\int_0^1 \sin\lambda_n \xi\, d\xi; \quad 1)\ d_n = \frac{2}{\pi n}\,[1 - (-1)^n], \quad 2)\ d_n = \frac{4}{\pi(2n-1)} \tag{2.3}$$

It follows from (2.3) that in the case of boundary conditions (1.2), corresponding to the clamped ends of the elastic element [case 1 in (1.8)], the motion of the carrier body has no effect on the even modes of oscillation $\Theta_{2k}(\theta)$; the coefficients of the sensitivity $d_{2k} = 0$. These modes are due solely to the initial distributions of the displacements and velocities, i.e. the coefficients φ_{2k}, ψ_{2k} $(k = 1, 2, \ldots)$. The required functions $\Theta_n(\theta)$ can be represented in the form of an integral operator of $\eta^{\cdot\cdot}(\theta)$ (of the acceleration of the carrier body)

$$\Theta_n(\theta, [\eta^{\cdot\cdot}]) = \varphi_n \cos\nu_n(\theta - \theta_0) + \frac{\psi_n}{\nu_n} \sin\nu_n(\theta - \theta_0) -$$

$$- \frac{d_n}{\nu_n} \int_{\theta_0}^{\theta} \eta^{\cdot\cdot}(\tau) \sin\nu_n(\theta - \tau)\, d\tau, \quad \Theta_n^{\cdot}(\theta, [\eta^{\cdot\cdot}]) = \frac{d\Theta_n}{d\theta} \tag{2.4}$$

Hence, a strong solution of boundary-value problems (1.8) exists and is unique if the function $\eta^{\cdot\cdot}(\theta)$ is continuous, and the series $\Sigma[(\nu_n\varphi_n)^2 + \psi_n^2]$ converges [1, 2]. According to (2.1) and (2.4) the solution can be represented in the form

$$z = z(\theta, \xi, [\eta^{\cdot\cdot}]) \equiv z_0(\theta, \xi) + z_\eta(\theta, \xi, [\eta^{\cdot\cdot}]) \tag{2.5}$$

$$z_0(\theta, \xi) \equiv \sum_{n=1}^{\infty} \sin\lambda_n\xi\,[\varphi_n \cos\nu_n(\theta - \theta_0) + \frac{\psi_n}{\nu_n}\sin\nu_n(\theta - \theta_0)]$$

$$z_\eta(\theta_0, \xi, [\eta^{\cdot\cdot}]) \equiv -\sum_{n=1}^{\infty} \frac{d_n}{\nu_n}\sin\lambda_n\xi \int_{\theta_0}^{\theta} \eta^{\cdot\cdot}(\tau)\sin\nu_n(\theta - \tau)\,d\tau$$

The term z_0 is due to the initial distributions φ, ψ, while z_η is due to the "external kinematic" action $\eta^{\cdot\cdot}(\theta)$.

Substituting (2.5) into (1.9) we obtain the required integro-differential Cauchy problem with respect to the unknown variable η, which is to be determined later. Note that a similar approach in reducing the initial problem to an integro-differential Cauchy problem for investigating oscillatory systems with lumped and distributed parameters was applied in [3–5] to a hybrid system with lumped and distributed parameters in the form of a rectangular vessel with a stably stratified liquid.

In a similar way we can consider the more-general problem when the elastic element is inhomogeneous: $\rho = \rho(x) \geqslant \rho_0 > 0$, $\sigma = \sigma(x) \geqslant \sigma_0 > 0$. In this case, it is required to construct systems of eigenvalues $\{\lambda_n\}$ and functions $\{\Xi_n(\xi)\}_{\rho^*(\xi)}$, orthogonal with weight $\rho^*(\xi)$, of the boundary-value problems corresponding to clamping conditions 1 and 2

$$\begin{gathered}(\sigma^*(\xi)\,\Xi')' + \lambda^2\rho^*(\xi)\,\Xi = 0, \quad 0 < \xi < 1 \\ 1)\ \Xi(0) = \Xi(1) = 0; \quad 2)\ \Xi(0) = \Xi'(1) = 0\end{gathered} \tag{2.6}$$

Here σ^*, ρ^* are the reduced characteristics of the elastic element. To derive the integro-differential Cauchy problem (1.9) it is necessary to obtain systems of eigenvalues and eigenfunctions of boundary value problems (2.6). There is a considerable literature devoted to constructing these, and powerful methods have been developed, which we obviously cannot review here. Further, the systems $\{\lambda_n\}$, $\{\Xi_n(\xi)_{\rho^*(\xi)}$ are assumed to be known; note that the eigenfrequencies $\nu_n = \lambda_n$, $n \geqslant 1$. When constructing the equation for $\eta(\theta)$ of the type (1.9) we must bear in mind that the term which takes into account the effect of the elastic element is equal to $\epsilon[\sigma^*(0)z'(\theta, 0) - \sigma^*(1)z'(\theta, 1)]$.

3. THE CONSTRUCTION OF A STANDARD INTEGRO-DIFFERENTIAL CAUCHY PROBLEM IN THE CASE OF A QUASI-LINEAR EXTERNAL FORCE AND A WEAK EFFECT OF THE ELASTIC ELEMENT

We will consider the integro-differential Cauchy problem (1.9) and (2.5), represented in explicit form

$$\begin{gathered}\eta^{\cdot\cdot} = \Pi(\theta, \eta, \eta^{\cdot}) + \epsilon\gamma(\theta) - \epsilon I(\theta, [\eta^{\cdot\cdot}]) \\ I(\theta, [\eta^{\cdot\cdot}]) \equiv \int_0^{\theta} E(\theta - \tau)\,\eta^{\cdot\cdot}(\tau)\,d\tau \\ \eta(\theta_0) = \eta^0, \quad \eta^{\cdot}(\theta_0) = \nu^0, \ \theta \in [\theta_0, \theta^*]\end{gathered} \tag{3.1}$$

Here the known function γ and the difference kernel E of the integral operator have the form

$$\begin{gathered}\gamma(\theta) \equiv z_0'(\theta, 0) - z_0'(\theta, 1) = \sum_{n=1}^{\infty} \lambda_n(1 - \cos\lambda_n)\,[\varphi_n \cos\nu_n(\theta - \theta_0) + \\ + \psi_n \nu_n^{-1}\sin\nu_n(\theta - \theta_0)], \quad 1)\ \cos\lambda_n = (-1)^n, \quad 2)\ \cos\lambda_n = 0 \\ E(\theta) = \sum_{n=1}^{\infty} e_n \sin\nu_n\theta, \quad e_n = d_n(1 - \cos\lambda_n), \quad E(0) = 0\end{gathered} \tag{3.2}$$

Note that series (3.2) can be summed and expressed in terms of the initial functions (and their derivatives) $\varphi(\xi)$, $\psi(\xi)$, specified in the range $\xi \in [0, 1]$ and oddly continued; they turn out to be

$2\pi/\nu_1$-periodic in θ. Moreover, it follows from (3.1) that when $\Pi=\gamma\equiv 0$ the motion of the rigid body (and the centre of mass of the system) will be uniform $\eta=\eta^0+\nu^0(\theta-\theta_0)$, $\theta\geqslant\theta_0$. If the external action Π depends only on θ, i.e. $\Pi=\Pi(\theta)$, Eq. (3.1) can be regarded as a linear integral equation of the Volterra type in the acceleration $w=\eta^{\bullet\bullet}$, having a unique solution $w(\theta)$ [6]. The required variables η, $\eta^\bullet$ are obtained by integrating $w(\theta)$, taking the initial conditions (3.1) into account. If $\Pi=\Pi(\theta,\eta^\bullet)$, then by introducing the variable $\nu=\eta^\bullet$ the order of the derivatives in the integro-differential Cauchy problem (3.1) can be reduced by one. In general, the integro-differential Cauchy problem (3.1) is equivalent to a system of three integral equations in η, v, w. Note also that in the cases when we can confine ourselves to considering a finite (usually small) number $N\geqslant 1$ of modes of oscillation of the elastic element and neglect in (3.2) terms with numbers $n>N$, the integro-differential Cauchy problem (3.1) can be reduced to a system of ordinary differential equations by differentiating with respect to θ and eliminating the integrals, provided the functions Π and j are smooth.

Below we investigate the integro-differential Cauchy problem (3.1), (3.2) assuming that the external action Π is the sum of a linear restoring elastic force and an arbitrary perturbation, periodic in θ (see Fig. 1)

$$\Pi(\theta,\vartheta,\eta,\eta^\bullet)=-\Omega^2(\eta-\eta_0)+\mu\Gamma(\omega\theta,\vartheta,\eta,\eta^\bullet) \tag{3.3}$$

Here $\Omega^2=(c/m)/(\sigma/\rho l^2)$ is a constant which has the meaning of the square of the reduced frequency, $c>0$ is the coefficient of elasticity of the restoring force, ω is the reduced frequency of the external action, μ is a small parameter $(0<\mu\leqslant\mu_0\ll 1)$, $\vartheta=\mu\theta$ is slow dimensionless time, and $\eta_0=\eta_0(\vartheta)$ is the position of equilibrium, which can be assumed to be zero. We will further assume that the value of the frequency Ω is of the same order of magnitude as the lowest eigenfrequencies of the elastic element ν_1, ν_2, etc., i.e. of the order of unity. The parameter $\epsilon=\rho l/m$ will also be assumed, like μ, to be small: $0<\epsilon\leqslant\epsilon_0\ll 1$. Note that in the limit when $\mu=\epsilon=0$, Eq. (3.1) describes linear oscillations of the rigid body, despite the singularities of the changes of variables (1.7) and expression (3.3).

By replacing the variables η, $v=\eta^\bullet$ by variables a, b of the Van der Pol type [3–5]

$$\begin{aligned}\eta&=\eta_0(\vartheta)+a\cos\Omega\theta+b\sin\Omega\theta\\ v&=\Omega(-a\sin\Omega\theta+b\cos\Omega\theta)\end{aligned} \tag{3.4}$$

we obtain a standard system in Bogolyubov form for the osculating variables a and b [7]

$$\begin{aligned}a^\bullet&=\mu\Gamma_a(\theta,\vartheta,a,b)+\epsilon\Omega^{-1}[-\gamma(\theta)+I]\sin\Omega\theta\\ b^\bullet&=\mu\Gamma_b(\theta,\vartheta,a,b)+\epsilon\Omega^{-1}[\gamma(\theta)-I]\cos\Omega\theta\\ a(\theta_0)&=a^0\equiv\eta^0-\eta_0(\vartheta_0),\quad b(\theta_0)=b^0\equiv\nu^0/\Omega\\ \Gamma_a&=-\Omega\eta_0'(\vartheta)\cos\Omega\theta-\Gamma(\omega\theta,\vartheta,\eta,v)\sin\Omega\theta\equiv\Gamma_a(\theta,\vartheta,a,b)\\ \Gamma_b&=-\Omega\eta_0'(\vartheta)\sin\Omega\theta+\Gamma(\omega\theta,\vartheta,\eta,v)\cos\Omega\theta\equiv\Gamma_b(\theta,\vartheta,a,b)\end{aligned} \tag{3.5}$$

In the expression for Γ the variables η, v are expressed in terms of θ, ϑ, a, b using Eqs (3.4). The function Γ, and together with it $\Gamma_{a,b}$ also, are doubly periodic, i.e. they have the frequency basis $\{\omega,\Omega\}$. System (3.5) is unsuitable for using asymptotic methods of averaging, since the integral operator I is defined by the functions $\eta^{\bullet\bullet}(\theta)$, for which there are no explicit formulae for replacing them in terms of the variables a, b, θ, ϑ ($\eta^{\bullet\bullet}$ can be expressed in terms of these and $a^\bullet$, $b^\bullet$). Attempts to use the formulae for integration by parts to obtain expressions in terms of lower derivatives (in terms of $\eta^\bullet$, η, see [3–5]) also meet certain difficulties, since the kernel $E(\theta)$ of the integral operator I is non-differentiable: $E(\theta)$ is a piecewise-continuous function while $E(\theta)$ is a generalized function, of the periodic impulse Dirac δ-function type [6]. Hence, we propose a more specific method, involving reducing the integro-differential Cauchy problem (3.5) to the form of a system of integral equations by integrating them with respect to θ using the formulae of repeated integration [3–5].

In fact, by integrating (3.5) with respect to θ and taking the initial conditions into account we obtain

$$a(\theta) = a^0 + \mu \int_{\theta_0}^{\theta} \Gamma_a(\tau, \kappa, a, b)d\tau - \epsilon\gamma_s(\theta) + \epsilon I_s(\theta, [v]) - \epsilon J_c(\theta, [v]), \quad \kappa = \mu\tau \tag{3.6}$$

$$b(\theta) = b^0 + \mu \int_{\theta_0}^{\theta} \Gamma_b(\tau, \kappa, a, b)d\tau + \epsilon\gamma_c(\theta) - \epsilon I_c(\theta, [v]) - \epsilon J_s(\theta, [v]), \quad v = \eta^{\cdot}$$

$$\gamma_{s,c}(\theta) = \frac{1}{\Omega}\int_{\theta_0}^{\theta} [\gamma(\tau) + b^0 E(\tau)] \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} \Omega\tau\, d\tau, \quad \theta \in [\theta_0, \theta^*]$$

$$I_{s,c} = \frac{1}{\Omega}\int_{\theta_0}^{\theta} v(\tau)E(\theta - \tau)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\Omega\tau\, d\tau, \quad J_{s,c} = \int_{\theta_0}^{\theta} v(\tau)\, d\tau \int_{\tau}^{\theta} E(\chi - \tau)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\Omega\chi\, d\chi$$

Here $I_{s,c}(\theta, [v])$, $J_{s,c}(\theta, [v])$ are integral operators of the variable $v(\theta)$, for which we have the representation (3.4), i.e. we finally obtain the operators of $a(\theta)$, $b(\theta)$. Hence, we have obtained a system of integral equations (3.6) in the unknowns $a(\theta)$, $b(\theta)$ in standard form [5, 7], corresponding to the integro-differential Cauchy problem (3.5). For a further approximate solution and analysis we can use the asymptotic approach of the method of averaging developed in [5] for the case of non-decaying periodic and almost (quasi-) periodic kernels of linear integral operators.

We will now construct the averaged integral equations and, on the basis of these, the averaged differential equations of the first approximation. The small parameters μ, ϵ are further assumed to be related. Using the averaging procedure [5] we will distinguish two qualitatively different modes of oscillation, corresponding to (a) the presence of internal resonance ($\Omega \simeq \nu_k$, where $k \geqslant 1$ is a "small" value of the index), and (b) there is no internal resonance ($\Omega \neq \nu_k$). These assumptions will be further refined in terms of the small parameters. Here the presence or absence of external resonance due to the external periodic action (with frequency ω) is permitted.

In conclusion, we emphasize the following extremely important point. The asymptotic solution of the Bogolyubov-standard system of the form $x^{\cdot} = \epsilon X(t, x)$, $x(t_0) = x^0$ reduces to the integration of the averaged system $\xi' = X_0(\xi)$, where X_0 is the average of X over t; $\xi(\tau_0) = x^0$, $\tau = \epsilon t$. From the point of view of the theory of integral equations, we can set up two systems

$$x(t) = x^0 + \epsilon \int_{t_0}^{t} X(t_1, x(t_1))dt_1, \quad \xi(\tau) = x^0 + \int_{\tau_0}^{\tau} X_0(\xi(\tau_1))d\tau_1$$

where $\tau = \epsilon t \sim 1$. In the case of an integro-differential Cauchy problem or an integral equation of the form

$$x^{\cdot}(t) = \epsilon \int_{t_0}^{t} X(t, t_1, x(t), x(t_1))dt_1, \quad x(t_0) = x^0$$

$$x(t) = x^0 + \epsilon \int_{t_0}^{t} Y(t, t_1, x(t), x(t_1))dt_1$$

averaging over the inner argument t_1 and justifying the closeness of the solutions of the initial and averaged equations are extremely problematical. In general, the average of the function Y over t_1 does not exist ("resonance" [3–5]). If it exists, it does not follow from the simplified equations obtained that x is a slow variable [5], and its determination does not become essentially simpler.

4. ASYMPTOTIC ANALYSIS OF THE OSCILLATIONS OF A HYBRID SYSTEM IN THE RESONANT AND NON-RESONANT CASES

We will consider different relationships between the small parameters μ and ϵ depending on the mode of oscillation [5].

4.1. *Averaged system in the resonant case*

The following relations are of interest in theory and practice

$$\begin{gathered}\Omega = \nu_k + O(\mu), \quad k = 1, 2, \ldots; \quad \epsilon = \mu^2 \\ \theta \in [\theta_0, \Theta/\mu], \quad \Theta = \text{const}\ (\theta^* \sim 1/\mu = 1/\sqrt{\epsilon})\end{gathered} \tag{4.1}$$

Relation (4.1) between μ and ϵ leads to the same order of magnitude of the perturbations [5]. Then the average values α, β of the variables a, b are described by two coupled integral equations [5] over a relatively narrow range of the slow time ϑ

$$\alpha(\vartheta) = a^0 + \int_{\vartheta_0}^{\vartheta} \Gamma_\alpha^*(\kappa, \alpha(\kappa), \beta(\kappa))\, d\kappa - \frac{e_k\nu_k}{4} \int_{\vartheta_0}^{\vartheta} (\vartheta - \kappa)\alpha(\kappa)\, d\kappa$$

$$\beta(\vartheta) = b^0 + \int_{\vartheta_0}^{\vartheta} \Gamma_\beta^*(\kappa, \alpha(\kappa), \beta(\kappa))\, d\kappa - \frac{e_k\nu_k}{4} \int_{\vartheta_0}^{\vartheta} (\vartheta - \kappa)\beta(\kappa)\, d\kappa \tag{4.2}$$

$$\vartheta = \mu\theta = \sqrt{\epsilon}\,\theta, \quad \vartheta \in [\vartheta_0, \Theta], \quad \kappa = \mu\tau$$

$$\Gamma_{\alpha,\beta}^*(\vartheta, \alpha, \beta) = \lim_{T\to\infty} \frac{1}{T} \int_{\theta_0}^{T} \Gamma_{a,b}(\theta, \vartheta, \alpha, \beta)\, d\theta$$

The remaining terms on the right-hand side of (3.6) (including the variable position of equilibrium $\eta_0 = \eta_0(\vartheta)$) make a contribution $O(\mu)$ to the solution, i.e. we have the estimates [5]

$$\begin{gathered}|a(\theta, \mu) - \alpha(\vartheta)| + |b(\theta, \mu) - \beta(\vartheta)| \leqslant C\mu \\ \theta \in [\theta_0, \Theta/\mu], \quad C = \text{const}\end{gathered} \tag{4.3}$$

Note that according to (3.5) there may be relations between the frequencies ω and Ω, leading to additional terms in the mean $\Gamma_{a,b}^*$ in (4.2). Further, by differentiating with respect to ϑ (twice), Eqs (4.2) are reduced to a coupled system of two second-order ordinary differential equations in the slow time ϑ, $\vartheta_0 \leqslant \vartheta \leqslant \Theta$

$$\begin{gathered}\alpha'' = \partial\Gamma_\alpha^*/\partial\vartheta + (\partial\Gamma_\alpha^*/\partial\alpha)\alpha' + (\partial\Gamma_\alpha^*/\partial\beta)\beta' - (e_k\nu_k/4)\alpha \\ \beta'' = \partial\Gamma_\beta^*/\partial\vartheta + (\partial\Gamma_\beta^*/\partial\alpha)\alpha' + (\partial\Gamma_\beta^*/\partial\beta)\beta' - (e_k\nu_k/4)\beta \\ \alpha(\vartheta_0) = a^0, \quad \alpha'(\vartheta_0) = \Gamma_\alpha^*(\vartheta_0, a^0, b^0) \\ \beta(\vartheta_0) = b^0, \quad \beta'(\vartheta_0) = \Gamma_\beta^*(\vartheta_0, a^0, b^0)\end{gathered} \tag{4.4}$$

For a number of classes of functions Γ (i.e. $\Gamma_{a,b}$), admitting of interpretation in terms of mechanics, system (4.4) can be completely integrated, in particular, if $\partial\Gamma_{\alpha,\beta}^*/\partial\alpha$, $\partial\Gamma_{\alpha,\beta}^*/\partial\beta$ are constants. If $\Gamma_\alpha^* = \Gamma_\alpha^*(\vartheta, \alpha)$, $\Gamma_\beta^* = \Gamma_\beta^*(\vartheta, \beta)$, Eqs (4.4) can be separated and, in the stationary case, topological and qualitative methods of investigation can be applied to them (phase-plane methods, see, for example, [8]). The Cauchy problem has been completely integrated [5] for the case when the perturbing non-linear additional term $\mu\Gamma$ in (3.3) defined a cubic non-linearity of the restoring force ($\Gamma = \Delta\eta^3$); the solution was obtained in terms of elliptic functions. Note also that the dynamic system (4.4) has a certain structure, the symmetry properties of which can be used to analyse it.

It is interesting to note that the perturbing effect of the integral term is equivalent to a linear "restoring force". Hence, when $\Gamma_{\alpha,\beta}^* \equiv 0$, the variables oscillate harmonically with frequency $\Lambda_k = (e_k\nu_k/4)^{1/2}$, and the required solution η, according to (3.4), is equal to ($\theta_0 = 0$)

$$\eta = \eta(\vartheta) + A\cos\mu\Lambda_k\theta\cos(\Omega\theta - \zeta) + O(\mu), \quad 0 \leqslant \theta \leqslant \Theta/\mu$$
$$v = \eta^{\cdot} = -A\Omega\cos\mu\Lambda_k\theta\sin(\Omega\theta - \zeta) + O(\mu) \tag{4.5}$$
$$\cos\zeta = a^0/A, \quad \sin\zeta = b^0/A, \quad A = (a^{0\,2} + b^{0\,2})^{1/2}$$

Hence, the oscillations of the rigid body perform beats with a low frequency $\mu\Lambda_k$ ($\mu = \sqrt{\epsilon}$) in dimensionless time θ, and the amplitude of the oscillations varies from a quantity of the order of unity to zero (quantities $O(\mu)$). It follows from (4.4) that the presence of an external periodic force $\Gamma = \Gamma(\omega\theta)$ in the resonant case ($\omega \simeq \Omega \simeq \nu_k$) does not lead to an "unlimited" linear increase in the quantities $\alpha(\vartheta)$, $\beta(\vartheta)$ due to the effect of the "restoring force", produced by the elastic element. In order that an additional slow "build up" should occur, it is necessary to modulate the action Γ, i.e. introduce a periodic dependence on ϑ: $\Gamma = \Gamma(\omega\theta, \Lambda_k\vartheta)$.

To reduce the system of integral equations (4.2) to the form of ordinary differential equations without assuming that $\Gamma^*_{\alpha,\beta}$ is differentiable with respect to ϑ, α, β, one can use the following appraoch. By differentiating system (4.2) with respect to ϑ once and introducing the new unknown variables p, q ($p' = \alpha$, $q' = \beta$), we obtain

$$p'' = \Gamma^*_\alpha(\vartheta, p', q') - \Lambda_k^2 p, \quad p(\vartheta_0) = 0, \quad p'(\vartheta_0) = a^0$$
$$q'' = \Gamma^*_\beta(\vartheta, p', q') - \Lambda_k^2 q, \quad q(\vartheta_0) = 0, \quad q'(\vartheta_0) = b^0 \tag{4.6}$$

Equations (4.6) differ considerably from (4.4) and are more compact in form. Together with (4.2) and (4.4) they make up the mathematical apparatus for investigating the oscillations of a hybrid system in the case of internal resonance.

4.2. *Averaged system in the non-resonant case*

The oscillations of the rigid body are described by an integral equation with the following natural assumptions

$$\Omega \neq \nu_k + O(\sqrt{\epsilon}) \quad (\Omega = \nu_k + O(1)), \quad k = 1, 2, \ldots; \quad \mu = \epsilon$$
$$\theta \in [\theta_0, \ \theta/\epsilon], \quad \Theta = \text{const}, \quad \vartheta = \epsilon\theta \tag{4.7}$$

External resonance, due to a periodic perturbing action of arbitrary frequency ω, as in Sec. 4.1, may or may not occur. Using the approach employed in [5], we obtain the following system of integral equations of the first approximation, corresponding to (3.6)

$$\alpha(\vartheta) = a^0 + \int_{\vartheta_0}^{\vartheta} \Gamma^*_\alpha(\kappa, \alpha(\kappa), \beta(\kappa))d\kappa + \Lambda \int_{\vartheta_0}^{\vartheta} \beta(\kappa)d\kappa$$
$$\beta(\vartheta) = b^0 + \int_{\vartheta_0}^{\vartheta} \Gamma^*_\beta(\kappa, \alpha(\kappa), \beta(\kappa))d\kappa - \Lambda \int_{\vartheta_0}^{\vartheta} \alpha(\kappa)d\kappa \tag{4.8}$$
$$\Lambda = \frac{\Omega}{2}\sum_{n=1}^{\infty}\frac{\nu_k e_k}{\nu_k^2 - \Omega^2}, \qquad \vartheta \in [\vartheta_0, \ \Theta]$$

The remaining terms in (3.6), including the terms in $\Gamma^*_{\alpha,\beta}$, due to the variability of the position of equilibrium $\eta_0(\vartheta)$ makes a contribution $O(\epsilon)$ for $\theta \sim \epsilon^{-1}$. By [5] we have the following estimate of the error between the solutions of the initial system of integral equations (3.6) and the averaged system (4.8)

$$|a(\theta, \epsilon) - \alpha(\vartheta)| + |b(\theta, \epsilon) - \beta(\vartheta)| \leqslant C\epsilon$$
$$\theta \in [\theta_0, \Theta/\epsilon], \quad C = \text{const} \tag{4.9}$$

Unlike system (4.2) [see (4.4) and (4.6)] corresponding to the case of internal resonance, when there is no resonance, the system of integral equations (4.8) is equivalent to two first-order differential equations, which follows from (4.8) after a single differentiation

$$\alpha' = \Gamma_\alpha^*(\vartheta, \alpha, \beta) + \Lambda\beta, \qquad \alpha(\vartheta_0) = a^0$$
$$\beta' = \Gamma_\beta^*(\vartheta, \alpha, \beta) - \Lambda\alpha, \qquad \beta(\vartheta_0) = b^0 \tag{4.10}$$

The main difference is that the effect of the integral term in (3.1) or (3.5) is equivalent to "gyroscopic forces". Note that the coefficient Λ can be of arbitrary sign and, in particular, can equal zero. For a number of classes of perturbing actions, having a mechanical meaning, Cauchy's problem (4.10) allows of a complete analytic and qualitative investigation. In particular, when (4.10) is stationary (time-independent), when $\Gamma_{\alpha,\beta}^* = \Gamma_{\alpha,\beta}^*(\alpha, \beta)$, phase-plane methods [7, 8] are applicable. If $\Gamma_{\alpha,\beta}^* = \text{const}$ (external resonance), the gyroscopic terms (like the restoring forces in Sec. 4.1) prevent an unlimited increase in α, β when $\Lambda \neq 0$; "build-up" in slow time is possible if the functions $\Gamma_{\alpha,\beta}^* = \Gamma_{\alpha,\beta}^*(\Lambda\vartheta)$ are periodic in ϑ. The case of non-resonant oscillations $\Gamma_{\alpha,\beta}^* \equiv 0$ is of interest in practice [4, 5]. By (3.4) and (4.10) we obtain the expressions ($\theta_0 = 0$)

$$\eta = \eta_0(\vartheta) + a^0 \cos(\Omega + \epsilon\Lambda)\theta + b^0 \sin(\Omega + \epsilon\Lambda)\theta + O(\epsilon)$$
$$\vartheta = -a^0 \Omega \sin(\Omega + \epsilon\Lambda)\theta + b^0 \Omega \cos(\Omega + \epsilon\Lambda)\theta + O(\epsilon) \tag{4.11}$$
$$\theta \in [0, \ \Theta\epsilon^{-1}], \quad \Theta = \text{const}$$

Hence, it follows that the effect of the elastic element reduces to a displacement of the frequency of oscillation of the unperturbed system by $\epsilon\Lambda$; there are no beats (see Sec. 4.1), and the amplitude is constant. The effect of a perturbing potential addition to the restoring force leads to a small, $O(\epsilon)$, change in the frequency by a constant amount [5]. System (4.10) has certain structural properties and symmetry, which can be used to analyse it.

In conclusion we note that the results obtained above can be employed to design systems of diagnostics (non-destructive testing) for analysing the functioning of an elastic element inside a closed cavity and inaccessible to direct observation.

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